Solving Iterated Functions Using Genetic Programming

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# Iterated Functions

<table>
<thead>
<tr>
<th>Iterated Function:</th>
<th>Answer:</th>
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<tbody>
<tr>
<td>$f(f(x)) = x$</td>
<td>$f(x) = x$</td>
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<tr>
<td>$f(f(x)) = x + 2$</td>
<td>$f(x) = x + 1$</td>
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<tr>
<td>$f(f(x)) = x^4$</td>
<td>$f(x) = x^2$</td>
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<tr>
<td>$f(f(x)) = (x^2 + 1)^2 + 1$</td>
<td>$f(x) = x^2 + 1$</td>
</tr>
<tr>
<td>$f(f(x)) = x^2 - 2$</td>
<td>$f(x) = ?$</td>
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</tbody>
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*Why is this problem so hard for humans?*
Test of Intelligence:

\[ f(f(x)) = x^2 - 2 \]

This problem has become famous in math and physics circles for requiring deep mathematical insight in order to solve.


Appeared in mathematical competitions

The rumored fastest solver
Michael Fisher
The *known* solution requires deep human insight to solve a special case. Assume \( f(f(x)) = g(a^2g^{-1}(x)) \):

\[
g(a^2g^{-1}(x)) = x^2 - 2
\]

Next assume \( a^2 = 2 \) and let \( \theta = g^{-1}(x) \):

\[
g(2\theta) = x^2 - 2,
\]
\[
g(2\theta) = g(\theta)^2 - 2,
\]
\[
x^2 - 2 = g(\theta)^2 - 2
\]

By inspection:

\[
x = g(\theta) = 2 \cos(\theta),
\]
\[
x = g(g^{-1}(\theta)) = 2 \cos(g^{-1}(\theta))
\]

Double angle formula:

\[
f(x) = 2\cos\left(\sqrt{2} \cos^{-1}\left(\frac{x}{2}\right)\right)
\]
But there are possibly many solutions

\[ f(f(x)) = x \]
\[ f(x) = x \]
\[ f(x) = -x \]
\[ f(x) = \frac{1}{x} \]

This a dark area of mathematics; Only a few special cases of functional problems have ever been solved.

Yet, Genetic Programming can find these solutions easily....
What is $f(x)$?

**Straightforward application of Symbolic Regression**

Fitness of a candidate $f(x) = -\frac{1}{n} \sum_{i=1}^{n} [ y_i - f(f(x_i)) ]^2$

Solutions iterated twice:
Solved in 81 seconds

And
Solved Reliably:

50 trials

50 trials
Nearly Perfect Fitness

\[ f(x) = \frac{16.4916 - 2 \cdot (1.16871 \cdot 10^{18}) \cdot x}{(1.16871 \cdot 10^{18}) \cdot x \cdot (16.4916 - 2 \cdot (1.16871 \cdot 10^{18}) \cdot x^2)} \]

The genetic program is trying to take a limit....

\[ f(x) = \lim_{a \to \infty} \frac{b - 2ax}{ax(b - 2ax^2)} \]
Exactly Correct Symbolically

\[ f(f(x)) = \frac{b - 2a(f(x))}{a(f(x))(b - 2a(f(x))^2)} \]

\[ \lim_{a \to \infty} f(f(x)) = \lim_{a \to \infty} \frac{b - 2a(f(x))}{a(f(x))(b - 2a(f(x))^2)} \]

\[ \lim_{a \to \infty} f(f(x)) = x^2 - 2 \]

The solution is symbolically correct
New Solution Found with Genetic Programming

\[ f(f(x)) = x^2 - 2 \]

\[ f(x) = \lim_{a \to \infty} \frac{1 - 2ax}{ax(1 - 2ax^2)} \]
Human Competitive:

- Long-developed and infamous problem in physics and mathematics
- Has required deep human insight into mathematics to solve special cases
- No other general method exists
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The Best Entry:

- Entirely new solution found via GP
- Fastest this problem has ever been solved
- Potential impact in many fields, where such problems have never been solved before
Conclusions

Use GP to Solve Iterated Functions

\[ f(f(f(x))) \]