Genetic Programming for Finite Algebras

— *HUMIES-2008* Presentation —

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Citation

Spector, Lee, Clark, David M., Lindsay, Ian, Barr, Bradford, and Klein, Jon. (2008). Genetic Programming for Finite Algebras. In *Proceedings of the Genetic and Evolutionary Computation Conference* (GECCO), edited by Maarten Keijzer et al. ACM Press. In press.

Everybody's Favorite Finite Algebra

$$\text{Boolean algebra, } \mathbf{B} := \langle \{0,1\}, \wedge, \vee, \neg \rangle$$

$\wedge$	0	1
0	0	0
1	0	1

$\vee$	0	1
0	0	1
1	1	1

	$\neg$
0	1
1	0

Primal: every possible operation can be expressed by a term using only (and not even) $\wedge$, $\vee$, and $\neg$.

Bigger Finite Algebras

- Have applications in many areas of science, engineering, mathematics
- Can be *much* harder to analyze/understand
- Number of terms grows astronomically with size of underlying set
- Under active investigation for decades, with major advances (cited fully in the paper) in 1939, 1954, 1970, 1975, 1979, 1991, 2008

Goal

- Find terms that have certain special properties
- *Discriminator* terms, determine primality

$$t^A(x,y,z) = \begin{cases} x \text{ if } x \neq y \\ z \text{ if } x = y \end{cases}$$

- *Mal'cev*, *majority*, and *Pixley* terms
- For decades there was no way to produce these terms in general, short of exhaustive search
- Current best methods produce enormous terms

Algebras Explored

$\mathbf{A}_1$ $*$	0	1	2
0	2	1	2
1	1	0	0
2	0	0	1

$\mathbf{A}_2$ $*$	0	1	2
0	2	0	2
1	1	0	2
2	1	2	1

$\mathbf{A}_3$ $*$	0	1	2
0	1	0	1
1	1	2	0
2	0	0	0

$\mathbf{A}_4$ $*$	0	1	2
0	1	0	1
1	0	2	0
2	0	1	0

$\mathbf{A}_5$ $*$	0	1	2
0	1	0	2
1	1	2	0
2	0	1	0

$\mathbf{B}_1$ $*$	0	1	2	3
0	1	3	1	0
1	3	2	0	1
2	0	1	3	1
3	1	0	2	0

Techniques

- Traditional genetic programming with ECJ
- Stack-based genetic programming with PushGP
- Alternative random code generators
- Asynchronous islands
- Trivial geography
- Parsimony-based selection
- Alpha-inverted selection pressure
- HAH = Historically Assessed Hardness

Results

- Discriminators for A_1, A_2, A_3, A_4, A_5
- Mal'cev and majority terms for B_1
- Example Mal'cev term for B_1:

$$(((((((((x*(y*x))*x)*z)*(z*x))*((x*(z*(x*(z*y))))*z))*z)*z)*(z*((((x*(((z*z)*x)*(z*x)))*x)*y)*(((y*(z*(z*y)))*(((y*y)*x)*z))*(x*(((z*z)*x)*(z*(x*(z*y))))))))))$$

Assessing Significance

Relative to prior methods:

- Uninformed search:
  - Exhaustive: analytical (expected value) and empirical search time comparisons
  - Random: analytical (expected value) and empirical search time comparisons
- Primality method: empirical term size comparisons

Significance, Time

	Uninformed Search Expected Time (Trials)
3 element algebras	
Mal'cev	5 seconds ($3^{15} \approx 10^7$)
Pixley/majority	1 hour ($3^{21} \approx 10^{10}$)
discriminator	1 month ($3^{27} \approx 10^{13}$)
4 element algebras	
Mal'cev	10^3 years ($4^{28} \approx 10^{17}$)
Pixley/majority	10^{10} years ($4^{40} \approx 10^{24}$)
discriminator	10^{24} years ($4^{64} \approx 10^{38}$)

Significance, Time

	Uninformed Search Expected Time (Trials)	GP Time
3 element algebras		
Mal'cev	5 seconds ($3^{15} \approx 10^7$)	1 minute
Pixley/majority	1 hour ($3^{21} \approx 10^{10}$)	3 minutes
discriminator	1 month ($3^{27} \approx 10^{13}$)	5 minutes
4 element algebras		
Mal'cev	10^3 years ($4^{28} \approx 10^{17}$)	30 minutes
Pixley/majority	10^{10} years ($4^{40} \approx 10^{24}$)	2 hours
discriminator	10^{24} years ($4^{64} \approx 10^{38}$)	?

Significance, Size

Term Type	Primality Theorem
Mal'cev	$10,060,219$
Majority	$6,847,499$
Pixley	$1,257,556,499$
Discriminator	$12,575,109$

(for A_l)

Significance, Size

Term Type	Primality Theorem	GP
Mal'cev	$10,060,219$	12
Majority	$6,847,499$	49
Pixley	$1,257,556,499$	59
Discriminator	$12,575,109$	39

(for A_1)

Criteria Satisfied

B: The result is equal to or better than a result that was accepted as a new scientific result at the time when it was published in a peer-reviewed scientific journal.

D: The result is publishable in its own right as a new scientific result independent of the fact that the result was mechanically created.

E: The result is equal to or better than the most recent human-created solution to a long-standing problem for which there has been a succession of increasingly better human-created solutions.

F: The result is equal to or better than a result that was considered an achievement in its field at the time it was first discovered.

G: The result solves a problem of indisputable difficulty in its field.

Human Competitive?

- Rather: human-**WHOMPING!**
- *Outperforms* humans *and all other known methods* on significant problems, providing benefits of *several orders of magnitude* with respect to search speed and result size
- Because there were no prior methods for generating practical terms in practical amounts of time, GP has provided the first solution to a previously open problem in the field

Potential Impact

These results are in an foundational area of pure mathematics with:

- A long history
- Many outstanding problems of theoretical significance and quantifiable difficulty
- Applications across the sciences

The Best Entry

Among the ways in which this is the **best** entry to the *2008 Human Competitive Results* competition:

- Numerical size of the benefit provided by evolutionary computation (up to 10^{14})
- Breadth of potential impacts and applications