Family of controllers based on sector non-linear functions: an application for first-order dynamical systems

Author(s): Marlen Meza-Sánchez ; Maria del Carmen Rodríguez-Liñán [1] ;
Eddie Clemente [2]
View affiliations

Source: IET Control Theory & Applications, 0pp.
DOI: 10.1049/iet-cta.2019.0680 , Print ISSN 1751-8644,
Online ISSN 1751-8652 Available online: 03 March 2020

Received 13/06/2019, Accepted 27/02/2020, Revised 17/12/2019, Published 02/03/2020

Article

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Inspec keywords: position control; asymptotic stability; control system synthesis; Lyapunov methods; robust control; nonlinear control systems; feedback

Other keywords: first-order dynamical systems; tracking control problems; nominal controllers; saturation bounds; constant bound value; control variable; sector nonlinear functions; traditional proportional controller

Subjects: Spatial variables control; Nonlinear control systems; Stability in control theory; Control system analysis and synthesis methods

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IET Control Theory & Applications

Brief Paper

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ISSN 1751-8644
Received on 13th June 2019
Revised 17th December 2019
Accepted on 27th February 2020
doi: 10.1049/iet-cta.2019.0680
www.ietdl.org

Marlen Meza-Sánchez, Maria del Carmen Rodríguez-Liñán[1], Eddie Clemente[2] ✉
[1]*Departamento de Estudios de Posgrado e Investigación, CONACYT-TecNM/I.T. Ensenada, Baja California, México*
[2]*Departamento de Estudios de Posgrado e Investigación, TecNM/ I. T. Ensenada, Baja California, México*
✉ *E-mail: eclemente@ite.edu.mx*

Abstract: This study proposes the design of a family of controllers based on sector non-linear functions for first-order dynamical systems. Three new controllers that incorporate these types of functions are presented and analysed to validate the authors' premise. The proposed nominal controllers and an augmented version with integral action are presented. Asymptotic stability is proven under the Lyapunov theory and the controllers' performance is compared against a traditional proportional controller. An empirically tuned relation depending on a constant bound value and an operation range is proposed; this is used to compute the gains of each controller. Simulation results with all of the controllers under saturation bounds are presented to illustrate the effectiveness of the method at solving the output regulation and the tracking control problems, under practical physical assumptions. The numerical comparison utilises the $\mathscr{L}_2$ and $\mathscr{L}_\infty$ norms over the output error, and over the control variable, applying the same saturation bounds for each controller.

1 Introduction

This work proposes the construction of non-linear controllers whose behaviour is comparable with that of a proportional (P) controller for first-order dynamical systems (FO-DSs). P, proportional–integral (PI), proportional–derivative (PD), and proportional–integral–derivative (PID) controllers are the most commonly used control schemes in industrial applications. Their success can be attributed to their simplicity and efficacy in obtaining the desired performance from a plant. A P-only control law applies a correction which is P to the error measured between the desired reference and the actual output of the plant. Even though P control is in itself incapable of guaranteeing zero steady-state error, unlike the PI and PID controllers, its simplicity makes it a regular choice for applications where the error's effect is negligible. Examples of such applications include inner-loop cascade control and surge tank level control applications [1, 2].

P control can be analysed as an odd, passive function, that lies in quadrants I and III of the Cartesian plane; its output is continuous and monotonically increasing. Sector bounded non-linear functions, in general, are non-linearities that reside in some sector of the plane. This sector can be conic, described by two straight lines, or it can be defined by less conservative bounds, given by some non-linear expression. This is known as a generalised sector bound [3]. In this work, we are concerned with non-linear functions within the sector $[0, \infty]$; for simplicity, we will refer to them as sector non-linear functions (SNFs), keeping in mind the sector in which we are interested. Under this consideration, one can say that the P controller is a sector function that belongs to the sector $[0, \infty]$.

This document poses the hypothesis that *it is possible to design stable non-linear controllers based on unbounded, continuous, and monotonically increasing functions lying in sector* $[0, \infty]$. In order to demonstrate this hypothesis, the passive properties of these SNFs are exploited in the construction of a Lyapunov function. This shows that the resulting closed-loop system is globally asymptotically stable. Then, it can be concluded that the controllers based on SNFs within sector $[0, \infty]$ present a stable closed-loop behaviour, which is comparable with that of a P-only controller. To exemplify our premise, three different SNF controllers are proposed for regulation and tracking of a FO-DS. Moreover, analogously to a PI controller, an integral action is added to the proposed SNF controllers to ensure the zero steady-state error.

FO-DSs are fundamental in different research areas such as robotics, telecommunications, chemistry, biology, sociology etc. [4]. For instance, in robotics, they are useful during robot design, analysis, control and simulation [5]. FO-DS can model the kinematic motion of rigid bodies; this is the robot motion within an operational space [6]. For example, in collaborative robotics FO-DS models constitute the simplest mathematical model for the description of agents (such as robots, satellites, autonomous robots etc.) [7]. FO-DSs are used in collaborative control of multi-robot systems during consensus and formation tasks. In [8, 9], this problem is addressed assuming that motion constraints for an agent, such as non-holonomic restrictions on mobile robots, are defined in their direction and magnitude. Another example can be found in [10], where both unbounded and bounded control laws are considered for centroid tracking and relative formation tasks. Group consensus between both first- and second-order discrete-time systems is considered in [11]. Utilising the matrix and graph theory, the authors define sufficient conditions to reach consensus in such an heterogeneous group. FO-DSs also appear in humanoid robotics, specifically during the modelling of motion components to generate walking patterns. In [12], a bounded relaxed condition to generate real-time walking patterns, called divergent component of movement (DCM), is proposed. Later, a simple first-order model for the DCM is presented by Hopkins *et al*. [13]. There, the authors introduce a method for reference trajectory planning of the DCM over uneven terrain, considering the trajectories of the vertical components of the centre of mass, and the zero-moment point. The DCM has also been extended to a three-dimensional version by Englsberg *et al*. in [14] to obtain a planning and control method for bipedal walking over uneven ground. The method takes advantage of the linear properties of DCM when modelled by a FO-DS.

Studies on PI, PD, PID controls and their variants have appeared over the years in several forms. To mention some examples, Katebi and Moradi [15] proposed the integration of some features of MPC controllers into the PID control problem by introducing a prediction horizon for a set of parallel PID controllers. An adaptive version of PID controllers using backstepping for linear second-order minimal phase processes is presented in [16]. Given its relevance to the field of control engineering, a great number of methods for PID tuning have been proposed in the literature over time. Take, for instance, the influential Ziegler–Nichols tuning rules [17] where the step and the frequency responses are used to obtain adequate gains for the P, I and D terms of the controller. A heuristic rule of thumb was proposed in [18] based on the well-accepted IMC-PID tuning rules for the industry. Some traditional widespread used model-based techniques are presented in [19–21]. Another model-based PID design tool for Matlab is introduced in [22] where some examples from the process industries are evaluated. Nguyen and Nguyen [23], for instance, present a methodology for the tuning of PID controllers for desired overshoot and settling time parameters. The procedure is demonstrated for first- and second-order dynamical systems. Another tuning strategy for PID control is proposed in [24] for a desired maximum sensitivity. The provided solution is an optimisation of the PID response over the exponential weighted error function. The computation of the stabilisation set for PI/PID controllers using root-counting and signature results for polynomials is reported in [25]. All of the above examples illustrate the relevance of the PID within the state of the art, and the remaining open problems that still exist, considering the extensive attention it has received from scholars since its conception. Thus, a comparison with a P controller is proposed in this paper.

The present work takes its inspiration in the results of [26, 27] to propose a family of controllers based on non-linear functions, which are not traditionally contemplated in the design of control laws. Some examples of these non-linear functions include a hyperbolic sine, a hyperbolic cosine, and a third degree polynomial (or cubic) function. The proposed controllers are compared against a typical P controller. All four controllers (hyperbolic sine, hyperbolic cosine, cubic, and P based) combine the inverse dynamics of the FO-DS. A version of the controllers incorporating the integral action is also analysed. Numerical results are given for the regulation and tracking problems in a FO-DS under hard and smooth saturation bounds representing the physical constraints in practical applications. The performance of the controllers is evaluated through the $\mathscr{L}_2$ and $\mathscr{L}_\infty$ norms. To achieve a fair comparison, the gains in each controller are empirically tuned via a proposed relation that characterises each of the functions. The gains are thus parameterised with respect to a defined saturation bound, and a given operation range for the error between the system output and the desired reference.

This document is organised in the following manner. The problem statement is presented in Section 2. Section 3 describes the design methodology for the SNF controllers and also includes stability proofs for the proposed family of controllers. In Section 4, the integral action is added to the controllers and the corresponding stability proof is also presented. Section 5 contains the proposed parameterisation relations for each controller, and numerical results for regulation and tracking tasks. Simulation results assuming a time-delayed system are presented in the same section. Finally, the conclusions of this work are given in Section 6.

1.1 Contribution

This work demonstrates that the hypothesis of using unbounded, continuous, and monotonically increasing SNFs derives in a family of suitable controllers. These kinds of controllers are capable of solving the output regulation and tracking control problems in FO-DSs. The main contributions of this paper are enumerated as follows:

i. A family of controllers, based on SNFs lying in sector $[0, \infty]$, is proposed. The performance of these controllers is comparable with the traditional input saturated P controller.
ii. Examples of SNF controllers using hyperbolic and cubic functions are given as particular cases to illustrate the design procedure and their behaviour under saturation bounds.
iii. Global asymptotic stability is demonstrated for: (a) the closed-loop system and (b) the closed-loop system plus integral action (resembling a typical PI controller), both under the Lyapunov stability criterion.
iv. The gains for each controller are chosen by introducing a relation between the gains and two tuning parameters. These parameters depend on the given saturation bound and a desired operation range. The gains are, thus, empirically tuned to achieve the desired performance.
v. Numerical comparisons using the $\mathscr{L}_2$ and $\mathscr{L}_\infty$ norms, over the output error and the over the control variable, are performed for each controller, under the same scenarios and applying the same saturation level.

2 Problem statement

Consider a FO-DS whose dynamics are described by the ordinary differential equation:

$$a\,\dot{q}(t) + b\,q(t) = c\,u(t), \tag{1}$$

where $q(t)$ and $u(t)$ are the output and the control input, respectively, and $\dot{q}(t)$ denotes the first time derivative of the output. The coefficients a, b, and c are the system's physical parameters.

The control objective can be described as the design of a control input variable $u(t)$ containing a continuous, monotonically increasing SNF lying in sector $[0, \infty]$, such that the output $q(t)$ asymptotically converges to the desired reference $q_{\rm d}(t)$, that is

$$\lim_{t \to \infty} \left| q_{\rm d}(t) - q(t) \right| = 0. \tag{2}$$

3 Synthesis of non-linear controllers with SNF

Definition 1 (Sector non-linearity): A function $\phi: \mathbb{R} \to \mathbb{R}$ is said to be in sector $[l, m]$ if for all $q \in \mathbb{R}$, $p = \phi(q)$ lies between lq and mq.

Remark 1: For a function $\phi(q)$ belonging to sector $[0, \infty]$, q and $\phi(q)$ always have the same sign. This is $q\phi(q) > 0$ for all q.
In this work, the use of SNF non-linear controllers addressing the tracking control problem in FO-DS is proposed. Let us define the tracking errors as

$$\tilde{q}(t) = q_{\rm d}(t) - q(t), \quad \dot{\tilde{q}}(t) = \dot{q}_{\rm d}(t) - \dot{q}(t), \tag{3}$$

and consider for (1) a redefinition of the control input $u(t)$ combining the inverse dynamics of the system and an SNF term. This is let $u(t)$ be given as

$$u_i(t) = \frac{a}{c}\big(\dot{q}_{\rm d}(t) + g^i_{\rm SNF}(\tilde{q}(t))\big) + \frac{b}{c}(q_{\rm d}(t) - \tilde{q}(t)), \tag{4}$$

where $g^i_{\rm SNF}(\tilde{q}(t))$ is either

$$g^1_{\rm SNF} = \sinh(k_1\tilde{q}), \tag{5}$$

$$g^2_{\rm SNF} = k_2\big(\tilde{q}^3 + \tilde{q}\big), \tag{6}$$

$$g^3_{\rm SNF} = \tilde{q}\cosh(k_3\tilde{q})\,. \tag{7}$$

In addition, k_i is the set of positive constant values for each control law $g^i_{\rm SNF}(\tilde{q}(t))$, for $i = 1, 2, 3$. As a side note, the controller using $g^1_{\rm SNF}(\tilde{q}(t))$ is inspired in [26, 27].

Remark 2: The proposed non-linear sector functions are odd functions within sector $[0, \infty]$, as stated in Definition 1, and they also fulfil Remark 1. That is $g^i_{\rm SNF}(\tilde{q}(t)) \in [0, \infty]$, then it holds that $\tilde{q}(t)g^i_{\rm SNF}(\tilde{q}(t)) > 0$, for all t.

The stability of the controlled system is demonstrated through the analysis of the closed-loop dynamics obtained by substituting (4) into (1).

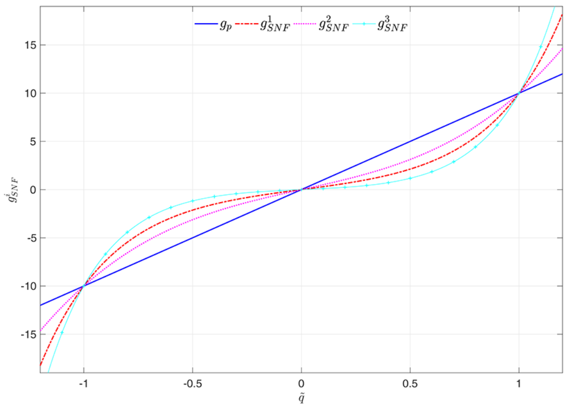

Fig. 1 *Graph of sector non-linear functions $g^i_{\mathrm{SNF}}(\tilde{q}(t))$, $i = 1, 2, 3$, and a P function g_p parameterised for $\eta = 1$ and $\gamma = 10$*

Theorem 1: The closed-loop system defined by the FO-DS (1) and the control input (4) is globally asymptotically stable.

Proof: The closed-loop dynamics is given by the first-order non-linear system

$$\dot{\tilde{q}}(t) + g^i_{\mathrm{SNF}}(\tilde{q}(t)) = 0. \tag{8}$$

Applying the classic quadratic Lyapunov candidate function $V(\tilde{q}) = (1/2)\tilde{q}^2 > 0$ yields $\dot{V}(\tilde{q}) = -\tilde{q}g^i_{\mathrm{SNF}}(\tilde{q}(t))$. It can be straightforwardly concluded that $\dot{V}(\tilde{q}) < 0$ (see Remark 2), with k_i defined as a set of positive constant values. Hence, the proof of Theorem 1 is concluded. □

4 Integral action

In order to guarantee a zero error in the steady state, the SNF controllers are augmented with an integral term. The analysis of such a case is presented below. Consider the errors $\tilde{q}(t)$ and $\dot{\tilde{q}}(t)$ defined as before and the previous set of three non-linear controllers given by (5)–(7). The augmented control input $u(t)$ is thus redefined as

$$\begin{aligned} u^{\mathrm{I}}_i(t) = {} & \frac{a}{c}\left(\dot{q}_{\mathrm{d}}(t) + g^i_{\mathrm{SNF}}(\tilde{q}(t)) + k^{\mathrm{I}}_i \int_0^t \tilde{q}(t)\mathrm{d}\tau\right) \\ & + \frac{b}{c}(q_{\mathrm{d}}(t) - \tilde{q}(t)), \end{aligned} \tag{9}$$

where k^{I}_i, $i = 1, 2, 3$, is a positive constant gain value for the integral term in each controller.

Remark 3: From Table 1, notice that the temporal derivatives $D_t[\, g^i_{\mathrm{SNF}}(\,\tilde{q}(t))]$ of the proposed non-linear sector functions $g^i_{\mathrm{SNF}}(\tilde{q}(t))$ have the form $\dot{\tilde{q}}(t)\bar{g}^i(\tilde{q}(t))$, where $\bar{g}^i(\tilde{q}(t))$ are positive definite functions.

Theorem 2: Consider the FO-DS (1) driven by the control input $u^{\mathrm{I}}_i(t)$ as in (9), where $g^i_{\mathrm{SNF}}(t)$ is a set of monotonically increasing non-linear functions in the sector $[0, \infty]$. Let $g^i_{\mathrm{SNF}}(t)$ be given either as (5), (6), or (7), and each k_i and k^{I}_i are positive constants. Then, the closed-loop system given by (1) and (9) is globally asymptotically stable.

Table 1 SNFs $g^i_{\mathrm{SNF}}(t)$ and their respective time derivatives

$g^i_{\mathrm{SNF}}(\tilde{q}(t))$	$D_t[g^i_{\mathrm{SNF}}(\tilde{q}(t))]$
$\sinh(k_1\tilde{q}(t))$	$\dot{\tilde{q}}(t)\{k_1\cosh(k_1\tilde{q}(t))\}$
$\tilde{q}(t)\cosh(k_2\tilde{q}(t))$	$\dot{\tilde{q}}(t)\{\cosh(k_2\tilde{q}(t)) + k_2\tilde{q}(t)\sinh(k_2\tilde{q}(t))\}$
$k_3(\tilde{q}^3(t) + \tilde{q}(t))$	$\dot{\tilde{q}}(t)\, k_3\{3\tilde{q}^2(t) + 1\}$

Proof: The closed-loop system is obtained as

$$\begin{aligned} \dot{\tilde{q}}(t) &= \dot{q}_{\mathrm{d}}(t) - \frac{c}{a}u^{\mathrm{I}}_i(t) - \frac{b}{a}q(t) \\ &= -\, g^i_{\mathrm{SNF}}(\tilde{q}(t)) - k^{\mathrm{I}}_i \int_0^t \tilde{q}(t)\mathrm{d}\tau\,. \end{aligned} \tag{10}$$

Then, differentiating once with respect to time, the closed-loop system becomes

$$\ddot{\tilde{q}}(t) = -\, D_t[g^i_{\mathrm{SNF}}(\tilde{q}(t))] - k^{\mathrm{I}}_i\tilde{q}(t), \tag{11}$$

where $D_t[g^i_{\mathrm{SNF}}(\tilde{q}(t))]$ is defined in Table 1.

Using the state-space notation, (11) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\tilde{q}(t)\\ \dot{\tilde{q}}(t)\end{bmatrix} = \begin{bmatrix}\dot{\tilde{q}}(t)\\ -D_t[g^i_{\mathrm{SNF}}(\tilde{q}(t))] - k^{\mathrm{I}}_i\tilde{q}(t)\end{bmatrix}. \tag{12}$$

Note that the closed-loop system has one unique equilibrium point at $\tilde{q}_{\mathrm{e}} = (0, 0)$. Now, let a Lyapunov candidate function be given as

$$V(\tilde{q}(t), \dot{\tilde{q}}(t)) = \frac{1}{2}[k^{\mathrm{I}}_i\tilde{q}^2(t) + \dot{\tilde{q}}^2(t)]\,. \tag{13}$$

Then, it can be shown that

$$\dot{V}(\tilde{q}(t), \dot{\tilde{q}}(t)) = -\,\dot{\tilde{q}}(t)D_t[g^i_{\mathrm{SNF}}(\tilde{q}(t))] < 0. \tag{14}$$

Furthermore, since $\tilde{q}_{\mathrm{e}} = (0, 0)$ is the only equilibrium point of system (12), then it can be concluded that the closed-loop system is globally asymptotically stable. The proof of Theorem 2 is thus finalised. □

5 Numerical example results

Numerical simulations are presented to illustrate the performance of the proposed SNF controllers under hard saturation constraints. They are compared against a P controller under hard and soft saturation bounds. The saturation constraint in the control input appears, in practice, as a physical bound for the demanded control achievable by the system. The saturated P controller is defined as u_{p} (for hard saturation) and u_{ht} (for soft saturation). The control laws u_{p} and u_{ht} in (4) are redefined using g_{p} (as the corresponding P controller) in place of g^i_{SNF}. The hard saturation function over a signal u is defined as $\mathrm{sign}(u)\min(|u|, \gamma)$ [28], while the soft saturation is implemented as $\gamma\tanh(u/\gamma)$ [29].

The procedure to define the gains to provide a fair comparison can be described as follows. A region of operation for the functions is selected in terms of the saturation bound γ and a chosen parameter η. This produces a slope in the P controller equal to γ/η. The rest of the controllers' gains are characterised by equating the functions to the P slope at the limits of such a region. This way, the functions' graphs present similar behaviour inside the region. This can be observed in Fig. 1.

The positive constant values are set to

$$k_1 = \eta^{-1}\mathrm{arsinh}\left(\frac{\gamma}{\eta}\right), \tag{15}$$

$$k_2 = \frac{\gamma}{\eta}(\eta^3 + \eta)^{-1}, \tag{16}$$

$$k_3 = \eta^{-1}\,\mathrm{arcosh}\left(\frac{\gamma}{\eta^2}\right), \tag{17}$$

respectively. The gain for the P controller is computed as $k_{\mathrm{p}} = \gamma/\eta$. In all cases, γ denotes the level of saturation. Notice the shape and decay of the functions within the range $\tilde{q} = [-\eta, \eta]$.

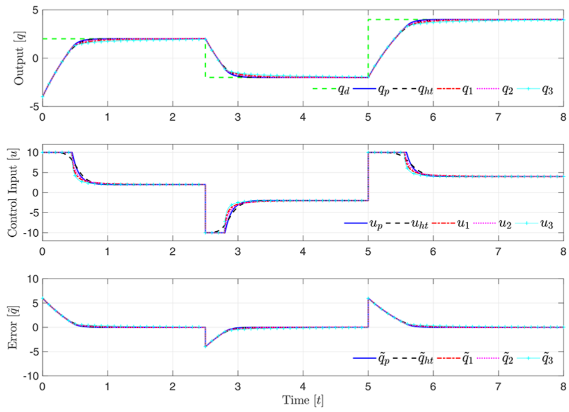

Fig. 2 *Performance comparison applying the saturated controllers in the form (4) for the output regulation problem. The subindexes indicate which controller is used, $i = 1, 2, 3$, for the sector non-linear functions $g^{i}_{\mathrm{SNF}}(t)$, g_p for the P controller under hard saturation, and g_{ht} for the P controller under soft saturation*

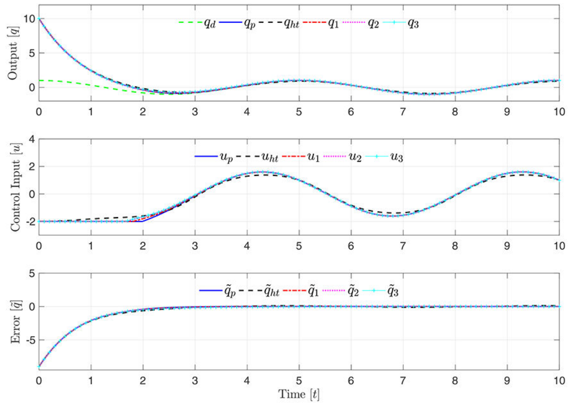

Fig. 3 *Performance comparison applying the saturated controllers in the form (4) for the tracking problem with a desired reference $q_d(t) = \cos((2\pi/5)t)$. Subindex $i = 1, 2, 3$, indicates the sector non-linear function g^{i}_{SNF} that is being used; subindexes p and ht stand for the P controller under hard and smooth saturation, correspondingly*

Consider the FO-DS (1) parameterised with unitary values for a, b, and c. For the output regulation problem, let the desired reference be a rectangular function given as

$$q_{\mathrm{d}}(t) = \begin{cases} 2, & 0 \le t \le 2.5, \\ -2, & 2.5 < t \le 5, \\ 4, & 5 < t\,. \end{cases} \tag{18}$$

In addition, let us propose a desired reference for the tracking control problem, described by

$$q_{\mathrm{d}}(t) = \cos\left(\frac{2\pi}{5}t\right), \tag{19}$$

Table 2 RMS of the error variable $\tilde{q}$ and control input u for the regulation problem in FO-DS. The controllers are applied using functions parameterised with $\gamma = 10$ and $\eta = 1$

Time period	$\mathscr{L}_2$	u_{p}	u_{ht}	u_1	u_2	u_3
[0, 1.5]	$\tilde{q}$	197.1408	197.9576	197.8610	197.4619	198.8121
	u	611.0703	598.5051	597.3484	603.1598	588.2772
[2.5, 3.5]	$\tilde{q}$	140.1047	141.6689	141.5336	140.7787	140.8394
	u	626.1572	607.5556	604.8654	614.1310	588.6660
[5, 6.5]	$\tilde{q}$	215.4392	216.2485	216.0072	215.6953	214.8570
	u	730.1944	716.9260	717.2607	723.4937	703.9441

$$\dot{q}_{\mathrm{d}}(t) = -\frac{2\pi}{5}\sin\left(\frac{2\pi}{5}t\right). \tag{20}$$

The performance of the controllers under study is evaluated through the $\mathscr{L}_2$-norm and the computation of the $\mathscr{L}_\infty$-norm. The $\mathscr{L}_2$-norm measures the RMS (root mean square) value of the argument, it is defined as

$$\mathscr{L}_2 = \sqrt{\frac{1}{t_{\mathrm{f}} - t_0}\int_{t_0}^{t_{\mathrm{f}}} \|\, \mathscr{I} \,\|^2 \, \mathrm{d}t}, \tag{21}$$

where $\mathscr{I}$ corresponds to the evaluated variable, and t_0 and t_{f} define the starting and finishing times of the interval over which the evaluation is made. The computation of the $\mathscr{L}_\infty$-norm over a period time of each simulation is done in accordance with

$$\mathscr{L}_\infty = \inf\{C \ge 0\colon |\mathscr{I}| \le C\}\,. \tag{22}$$

Furthermore, the amount of time during which the control input of each controller remains in saturated mode is also calculated. The case when the saturated controllers are applied to a first-order time-delayed system is also presented. All the comparison criteria are computed over the output error $\tilde{q}$ and over the control input u, considering a sampling time $t_m = 1 \times 10^{-4}$ s.

5.1 Saturated controllers

Let the system be driven by controllers $u_i(t)$ given as in (4), with the proposed SNFs, as well as by the P controller. Let $\eta = 1$ and $\gamma = 10$; thus, $k_1 = 2.9982$, $k_2 = 5$, $k_3 = 2.9932$, and $k_{\mathrm{p}} = 10$. Notice that γ is the level of saturation for all the controllers.

Numerical results are presented in Fig. 2 with initial condition $q(0) = -4$. Here, $|\tilde{q}(0)| > \eta$ was employed to show the output of the FO-DS with $g^{i}_{\mathrm{SNF}}(t)$, g_{p} and g_{ht} greater than γ. Each considered controller is able to solve the output regulation problem without significant differences regarding the convergence time.

For the tracking control problem, values of $\eta = 1$ and $\gamma = 2$ are selected; thus one has $k_1 = 1.4436$, $k_2 = 1$, $k_3 = 1.317$, and $k_{\mathrm{p}} = 2$. Fig. 3 shows the convergence of the system to the desired reference (top), the applied control input (centre), and the tracking error (bottom). The SNF controllers reach the reference as fast as the P control under hard saturation with a slightly lower demand of energy. Notice that when a soft saturation is used for the P controller, less energy is required to reach the reference, however the tracking error is larger in this case.

5.1.1 Performance comparison: The $\mathscr{L}_2$ norms calculated for the regulation problem applying all the proposed controllers under comparison are presented in Table 2. To independently analyse the different scenarios in the simulation, the norm is computed along three time intervals. This is for $t \in [0, 1.5]$, $t \in [2.5, 3.5]$, and $t \in [5, 6.5]$ s. The $\mathscr{L}_\infty$-norm is computed for the steady-state response of each controller and results are shown in Table 3 over the time intervals $t \in [2.0, 2.5]$, $t \in [4.5, 5]$, and $t \in [7.5, 8]$ s. The time that each controller operates in the saturated mode during the transient response is presented in Table 4 for the time intervals $t \in [0, 2.5]$, $t \in [2.5, 5]$, and $t \in [5, 8]$ s.

According to our previous qualitative assessment, the proposed controllers using the non-linear sector functions demand less energy compared with the traditional P controller under hard saturation. When soft saturation is used, this is with u_{ht}, the P demands less energy than the cubic controller in all three scenarios than the hyperbolic sine in one. However, the soft saturation P also presents the largest error. This is true for the three SNFs considered, except for u_3 over $t \in [0, 1.5]$. Taking the average RMS values of each controller during the transients, one can see that the SNF controllers, and in particular, the hyperbolic cosine-based controller, demand less energy than the P controller under hard saturation. The P controller with soft saturation, labelled u_{ht}, presents the greatest error overall. The RMS comparison of the output error $\tilde{q}$ places the performance of the SNF-based controllers between the results obtained from the hard and soft saturated P.

Also for the regulation problem, the steady-state response of each controller is analysed in terms of the $\mathscr{L}_\infty$-norm. From Table 3, one can see that the proposed SNF controllers exhibit larger values for the norm calculated over the error $\tilde{q}$ when compared against the P. Among them, the cubic is the one with the lowest norm value over the error. These results are reasonable since the proposed SNF controllers have a faster decay, meaning that their response is smoother when compared with the P controller. The amount of energy consumed by all the controllers is comparable, with the lowest demand obtained with the P bounded by smooth saturation. Finally, Table 4 shows the total amount of time that each controller stays in the saturation mode. Note that, because of the way that the smooth saturation function is defined, the u_{ht} controller does not reach the saturation limits, giving a total saturated time of 0. For the rest of the considered controllers, each of the SNF remains saturated for a slightly shorter period of time in contrast with u_p. Also, the cosh-based controller u_3 presents the shortest saturation times.

A similar analysis is realised for the tracking case. The computed $\mathscr{L}_2$- and $\mathscr{L}_\infty$ norms for each controller are presented in Table 5. The $\mathscr{L}_2$-norm is computed during the transient time, this is for $t \in [0, 4]$ s, while the $\mathscr{L}_\infty$-norm is calculated over the last 2 s of simulation. Notice that the behaviour of the system is consistent with the results from the regulation problem. The $\mathscr{L}_2$-norm over the error is similar to the SNF controllers and the hard saturation P. Conversely, u_{ht} presents the highest error. In terms of energy consumption, the SNF controllers demand less energy than the hard saturated P controller. Again, u_{ht} is the controller that demands the less energy, which is natural. One could rank the considered SNF controllers from lowest to highest demand of energy. Then, the cosh-based controller would come in first, then the cubic polynomial-based controller, and finally, the one using the hyperbolic sine. Similar values are obtained from computed $\mathscr{L}_\infty$-norm as shown in Table 5. In this case, u_{ht} presents the highest value for the norm over the error, and consequently, the energy consumed by u_{ht} is the lowest, as can be seen from the $\mathscr{L}_\infty$-norm values.

As before, the total time that each controller remains in saturated mode, during the tracking task, is calculated. Results are presented in Table 6. As expected, the P under soft saturation does not saturate at all. The SNF controllers, however, spend less time on saturated mode than u_p. These results are consistent with those obtained from the regulation problem.

Considering the outcomes of the regulation and tracking problems, it can be observed that, in average, the compared controllers offer similar behaviours since their gains were tuned under analogous criteria. This implies that one can choose the controllers' parameters according to the problem's particular needs, i.e. to reduce the error or the energy consumption.

Table 3 $\mathscr{L}_\infty$-norm of the error variable $\tilde{q}$ and control input u for the regulation problem in FO-DS. The controllers are applied using functions parameterised with $\gamma = 10$ and $\eta = 1$

Time period	$\mathscr{L}_\infty$	u_p	u_{ht}	u_1	u_2	u_3
[2, 2.5]	$\tilde{q}$	1.7056e−07	0.0027	0.0056	2.9203e−04	0.0855
	u	2.0000	1.9973	2.0113	2.0012	2.0028
[4.5, 5]	$\tilde{q}$	3.6520e−08	0.0027	0.0036	1.3516e−04	0.0727
	u	2.0000	1.9973	2.0071	2.0005	2.0017
[7.5, 8]	$\tilde{q}$	3.2700e−09	0.0232	0.0017	3.9722e−05	0.0566
	u	4.0000	3.9768	4.0034	4.0020	4.0008

Table 4 Elapsed time of the control input u operating in saturated mode for the regulation problem in FO-DS. The controllers are applied using functions parameterised with $\gamma = 10$ and $\eta = 1$

Time period	u_p	u_{ht}	u_1	u_2	u_3
[0, 2.5]	0.4543	0.0000	0.4459	0.4480	0.4449
[2.5, 5]	0.3001	0.0000	0.2917	0.2938	0.2865
[5, 8]	0.5878	0.0000	0.5568	0.5655	0.5486

Table 5 Performance evaluation for the tracking control of reference (19) in FO-DS. The controllers are applied using functions parameterised with $\gamma = 2$ and $\eta = 1$

Time period	$\mathscr{L}_2$	u_p	u_{ht}	u_1	u_2	u_3
[0, 4]	$\tilde{q}$	267.6464	269.6740	267.7508	267.9398	267.9576
	u	161.4933	147.7589	159.2578	156.8765	156.6727
Time period	$\mathscr{L}_\infty$	u_p	u_{ht}	u_1	u_2	u_3
[8, 10]	$\tilde{q}$	$3.0745e^{-05}$	0.1206	$3.5322e^{-05}$	0.0010	0.0011
	u	1.6060	1.3823	1.6060	1.6060	1.6060

Table 6 Time in saturated mode for the tracking control problem. The controllers are applied using functions parameterised with $\gamma = 2$ and $\eta = 1$

Time period	u_p	u_{ht}	u_1	u_2	u_3
[0, 10]	1.9862	0.0000	1.8054	1.6891	1.6741

5.1.2 Comparison under input time delay: A new simulation was implemented to evaluate the performance of the proposed controllers for the FO-DS (1) with an input time delay $\tau = 0.5$ s. The controllers' parameters are manually tuned and the desired reference is now modified to account for the increased settling time for all of the controllers. The same level of saturation is used, i.e. $\gamma = 10$. The gains of the P and the SNF controllers are computed with the new value of $\eta = 2.8$, giving $k_1 = 0.7090$, $k_2 = 0.1443$, $k_3 = 0.2594$, and $k_p = 3.5714$. The numerical results are shown in Fig. 4. In this case, the cosh controller provides a smooth response without overshoot and reaches the steady-state faster. Both the sinh and the cubic controllers produce a smaller overshoot than the P controller in the output's response. Additionally, the P control, under hard and smooth saturation, exhibits oscillations and takes longer time to reach the reference. Notice that the parameter η serves as a tuning value that adjusts the velocity of the system's response, and an increase in η results in a decreased gain for every controller.

6 Conclusions

The present document studies the design of new control schemes based on SNFs for FO-DSs. Given the properties of continuous monotonically increasing SNFs lying in sector $[0, \infty]$, it is possible to prove the global asymptotic stability of the closed-loop system.

The proposed controllers differ from those found in the literature in that they are classified as unbounded, continuous, monotonically increasing, and non-linear. To exemplify the proposed methodology, three controllers are built based on sector non-linear functions satisfying the above properties. The chosen functions are a hyperbolic sine, a hyperbolic cosine, and a cubic polynomial. Furthermore, the global asymptotic stability of the SNF controllers augmented with integral action is demonstrated as well.

The performance evaluation and validity of the proposed controllers is realised through numerical simulation and via computation of the $\mathscr{L}_2$ and $\mathscr{L}_\infty$ norms over the output errors and the control signals. The SNF controllers are compared against the traditional P controller under hard and soft saturation bounds. The P controller is classical and well studied within the control theory framework. The obtained numerical results show that all the controllers offer a similar behaviour. The total amount of time

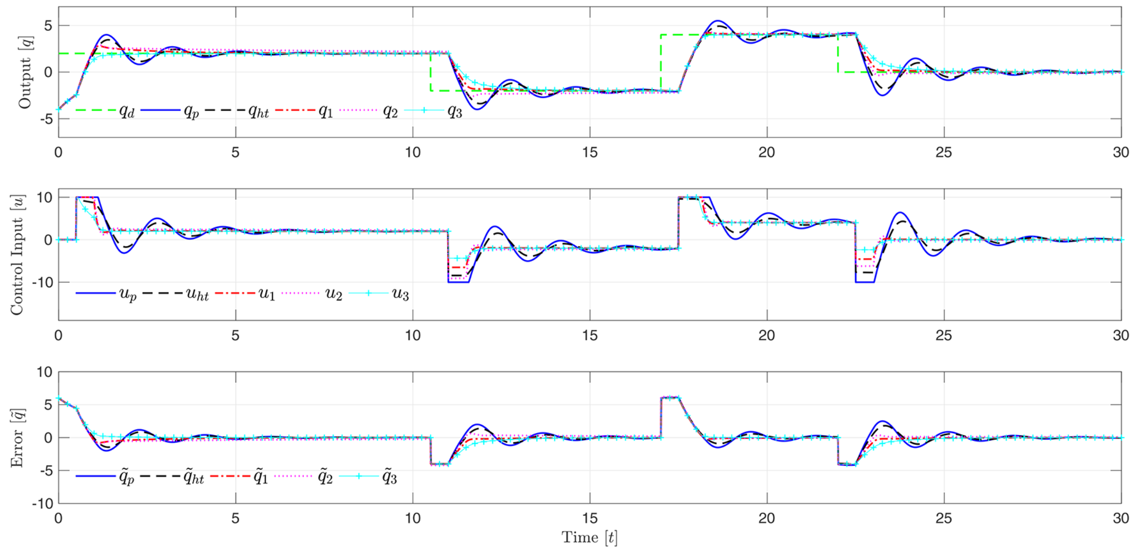

Fig. 4 *Performance comparison of the tuned controllers applied to the FO-DS with input time delay, with* $\tau = 0.5$ s*, using* $\eta = 2.8$ *and* $\gamma = 10$

during which the controllers stay in saturated mode is also computed; the SNF controllers present shorter amount of times under saturation than the P controller with hard saturation. As an illustrative example, the case of the saturated FO-DS with time delay is also considered. It is shown that the proposed SNF controllers satisfy the regulation objective comparatively better than the traditional P controller. The controllers were tuned to obtain a fair comparison among them, rather than to satisfy specific performance criteria. As such, the selection of the gains presented here should not be considered an optimal tuning method. Further research is needed to determine a methodology to tune the controllers for specific behaviour objectives (damping, settling time etc.).

The unbounded growth featured in the proposed functions would normally be considered a disadvantage in control theory; however, it should be indicated that these functions are passive and their global stability is demonstrable by Lyapunov. Moreover, their decay can aid in rapidly decreasing the control input, thus reducing their consumption of energy. This is an important fact since, in practice, control systems operate under saturation bounds due to physical restrictions inherent to the system.

7 Acknowledgements

The authors wish to thank CONACYT and Tecnológico Nacional de Mexico/I.T. Ensenada for their support given for the realisation of this paper, through project Cátedras 923.

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