

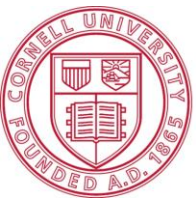
$$f(f(x))$$

Solving Iterated Functions Using Genetic Programming

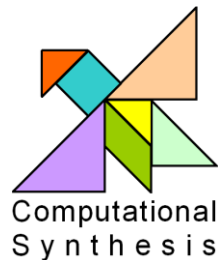
Michael Schmidt

Hod Lipson

2010 HUMIES Competition



Cornell University



Iterated Functions

Iterated Function:

$$f(f(x)) = x$$

$$f(f(x)) = x + 2$$

$$f(f(x)) = x^4$$

$$f(f(x)) = (x^2 + 1)^2 + 1$$

$$f(f(x)) = x^2 - 2$$

Answer:

$$f(x) = x$$

$$f(x) = x + 1$$

$$f(x) = x^2$$

$$f(x) = x^2 + 1$$

$$f(x) = ?$$

Why is this problem so hard for humans?

Test of Intelligence:

$$f(f(x)) = x^2 - 2$$

This problem has become famous in math and physics circles for requiring deep mathematical insight in order to solve.



"Mathvn journal problems," in Mathvn. vol. 01/2009 mathvn.org, 2009.

Appeared in mathematical competitions



B. A. Brown, A. R. Brown, and M. F. Shlesinger, "Solutions of Doubly and Higher Order Iterated Equations," Journal of Statistical Physics, vol. 110, pp. 1087-1097, 2003.

**The rumored fastest solver
Michael Fisher**

The *known* solution requires deep human insight to solve a special case

Assume $f(f(x)) = g(a^2 g^{-1}(x))$:

$$g(a^2 g^{-1}(x)) = x^2 - 2$$

Next assume $a^2 = 2$ and let $\theta = g^{-1}(x)$:

$$g(2\theta) = x^2 - 2,$$

$$g(2\theta) = g(\theta)^2 - 2,$$

$$x^2 - 2 = g(\theta)^2 - 2$$

By inspection:

$$x = g(\theta) = 2 \cos(\theta),$$

$$x = g(g^{-1}(\theta)) = 2 \cos(g^{-1}(\theta))$$

Double angle formula:

$$f(x) = 2 \cos\left(\sqrt{2} \cos^{-1}\left(\frac{x}{2}\right)\right)$$

But there are possibly many solutions

$$f(f(x)) = x$$

$$f(x) = x$$

$$f(x) = -x$$

$$f(x) = 1/x$$

This a dark area of mathematics; Only a few special cases of functional problems have ever been solved.

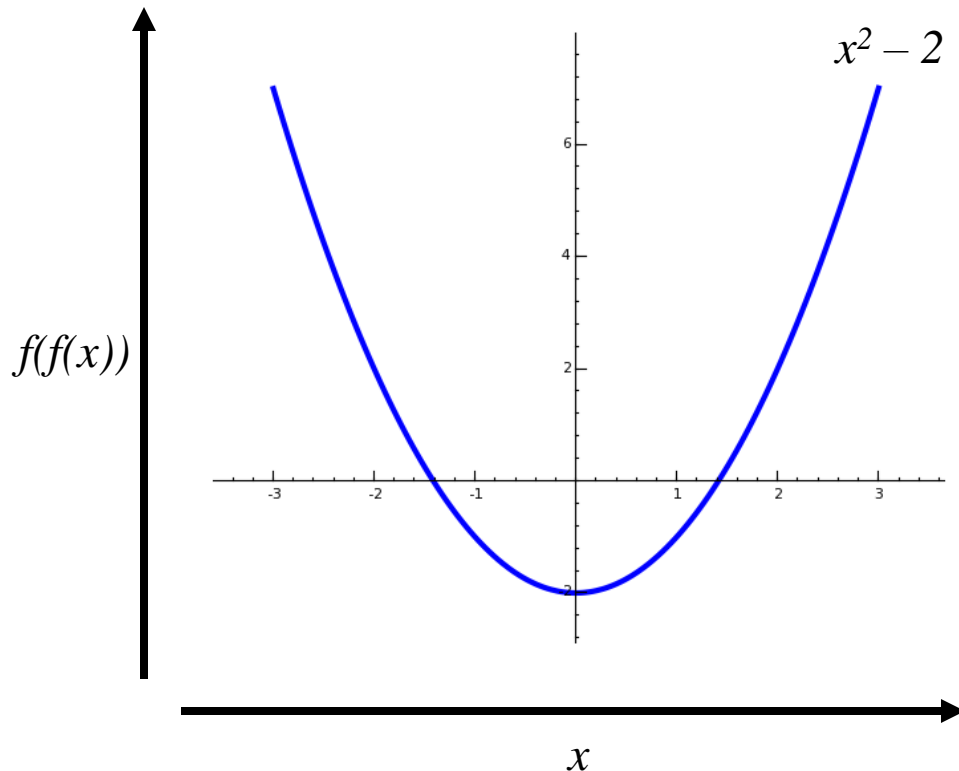
Yet, Genetic Programming can find these solutions easily....

$$f(f(x)) = x^2 - 2$$

Straightforward application of Symbolic Regression

$$\text{Fitness of a candidate } f(x) = -\frac{1}{n} \sum_{i=1}^n [y_i - f(f(x_i))]^2$$

Solutions iterated twice:



What is $f(x)$?

Solved in 81 seconds

Search Relation:
Find for a formula ($f(x)$) to satisfy...
 $y = f(f(x))$

Fitness metric:
Mean Absolute Error (Default)

Time variable:
x (radians)

Weight the errors by:
x (radians)

Using building blocks:

Operation	Format
<input type="checkbox"/> constant	1.234
<input checked="" type="checkbox"/> Add	$x + y$
<input type="checkbox"/> Subtract	$x - y$
<input checked="" type="checkbox"/> Multiply	$x * y$
<input checked="" type="checkbox"/> Divide	x / y
<input type="checkbox"/> Square root	\sqrt{x}
<input type="checkbox"/> Exponential	$\exp(x)$
<input type="checkbox"/> Logarithm	$\log(x)$
<input type="checkbox"/> Sine (radians)	$\sin(x)$
<input type="checkbox"/> Cosine (radians)	$\cos(x)$
<input type="checkbox"/> Tangent (radians)	$\tan(x)$

Use the following computers:

Server	Hostname
<input checked="" type="checkbox"/>	127.0.0.1
<input type="checkbox"/>	earthrus-1.mae.cornell.edu
<input type="checkbox"/>	128.253.249.183
<input type="checkbox"/>	128.253.249.247
<input type="checkbox"/>	ec2-174-129-158-91.compute-1.amazonaws.com
<input type="checkbox"/>	ec2-75-101-219-183.compute-1.amazonaws.com
<input type="checkbox"/>	ec2-72-44-38-139.compute-1.amazonaws.com

Run the formula search: Start Pause End

Progress and performance statistics:

Statistic	Value	Rate
Search Duration	2m 21s	
Total Core Hours	0.31	
Connected Servers	1/1	
Total Cores	8	
Formula Evaluations	2.965	20.42M e/s
Population Size	512	
Solution Generations	21665	160 g/s
Predictor Generations	21665	160 g/s
Trainer Generations	216.65	1.60 g/s
Mean Error (Best)	4.8e-5	
Mean Age (Best)	196.78	
Mean Size (Best)	31	
Mean Error (Current)	4.8e-5	
Mean Age (Current)	196.78	
Mean Size (Current)	31	

The best solution error over the search time

Evaluations/second

Generations/second

List of current solutions

Size	FR	Solution
15	2.000	$f(x) = -2.903x^2 - 2.22e-014$
13	4.843	$f(x) = 0.00887x^2 + 0.343$
11	4.849	$f(x) = \frac{0.00421}{x} + 0.343$
7	8.878	$f(x) = 3.92 + 0.192x^2$
5	18.312	$f(x) = 0.227x^2$
1	50.629	$f(x) = 48.0$

Selected solution plotted with the data

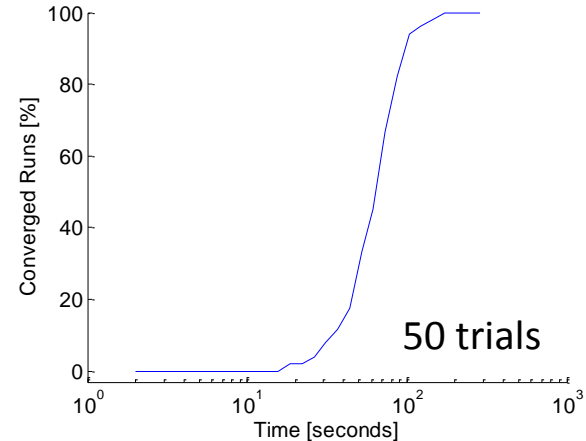
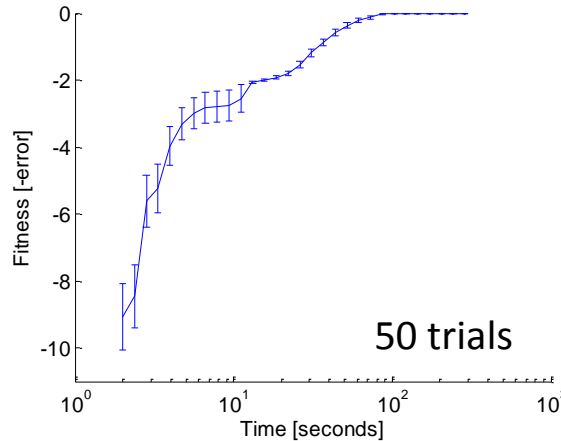
Quick statistics of solution:

$-2x^2 - 2.22e-014$
 $x_0 = 9e+013, x_1 = 0$

Name	Train Data	Validation Data
Sample Size	1408	1408
Fitness	-2.000001	-2.000001
R-squared	0.999946	0.999932
Correlation Coeff	0.999974	0.999968
AIC	-4242.024	-3908.789429
Mean Squared Error	0.048118	0.060955
Mean Absolute Error	0.013938	0.042176

Accuracy/complexity front of best solutions

And Solved Reliably:



Nearly Perfect Fitness

$$f(x) = \frac{16.4916 - 2 \cdot (1.16871 \cdot 10^{18}) \cdot x}{(1.16871 \cdot 10^{18}) \cdot x \cdot (16.4916 - 2 \cdot (1.16871 \cdot 10^{18}) \cdot x^2)}$$

The genetic program is trying to take a limit....

$$f(x) = \lim_{a \rightarrow \infty} \frac{b - 2ax}{ax(b - 2ax^2)}$$

Exactly Correct Symbolically

$$f(f(x)) = \frac{b - 2a(f(x))}{a(f(x))(b - 2a(f(x)))^2}$$

$$\lim_{a \rightarrow \infty} f(f(x)) = \lim_{a \rightarrow \infty} \frac{b - 2a(f(x))}{a(f(x))(b - 2a(f(x)))^2}$$

$$\lim_{a \rightarrow \infty} f(f(x)) = x^2 - 2$$

The solution is symbolically correct

New Solution Found with Genetic Programming

$$f(f(x)) = x^2 - 2$$



$$f(x) = \lim_{a \rightarrow \infty} \frac{1 - 2ax}{ax(1 - 2ax^2)}$$

Human

Competitive:

- Long-developed and infamous problem in physics and mathematics
- Has required deep human insight into mathematics to solve special cases
- No other general method exists

Human Competitive:

- Long-developed and infamous problem in physics and mathematics
- Has required deep human insight into mathematics to solve special cases
- No other general method exists

The Best Entry:

- Entirely new solution found via GP
- Fastest this problem has ever been solved
- Potential impact in many fields, where such problems have never been solved before

Conclusions

Use GP to Solve Iterated Functions

$$f(f(x))$$