$$f(f(x))$$

Solving Iterated Functions Using Genetic Programming

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Computational Synthesis

Iterated Functions

Iterated Function:	**Answer:**
$f(f(x)) = x$	$f(x) = x$
$f(f(x)) = x + 2$	$f(x) = x + 1$
$f(f(x)) = x^4$	$f(x) = x^2$
$f(f(x)) = (x^2 + 1)^2 + 1$	$f(x) = x^2 + 1$
$f(f(x)) = x^2 - 2$	$f(x) = ?$

Why is this problem so hard for humans?

Test of Intelligence:

$$f(f(x)) = x^2 - 2$$

This problem has become famous in math and physics circles for requiring deep mathematical insight in order to solve.

"Mathvn journal problems," in Mathvn. vol. 01/2009 mathvn.org, 2009.

Appeared in mathematical competitions

B. A. Brown, A. R. Brown, and M. F. Shlesinger, "Solutions of Doubly and Higher Order Iterated Equations," Journal of Statistical Physics, vol. 110, pp. 1087-1097, 2003.

The rumored fastest solver
Michael Fisher

The *known* solution requires deep human insight to solve a special case

Assume $f(f(x)) = g(a^2 g^{-1}(x))$:

$$g(a^2 g^{-1}(x)) = x^2 - 2$$

Next assume $a^2 = 2$ and let $\theta = g^{-1}(x)$:

$$g(2\theta) = x^2 - 2,$$

$$g(2\theta) = g(\theta)^2 - 2,$$

$$x^2 - 2 = g(\theta)^2 - 2$$

By inspection:

$$x = g(\theta) = 2\cos(\theta),$$
$$x = g(g^{-1}(\theta)) = 2\cos(g^{-1}(\theta))$$

Double angle formula:

$$f(x) = 2\cos\left(\sqrt{2}\cos^{-1}\left(\frac{x}{2}\right)\right)$$

But there are possibly many solutions

$$f(f(x)) = x$$

$$f(x) = x$$

$$f(x) = -x$$

$$f(x) = 1/x$$

This a dark area of mathematics; Only a few special cases of functional problems have ever been solved.

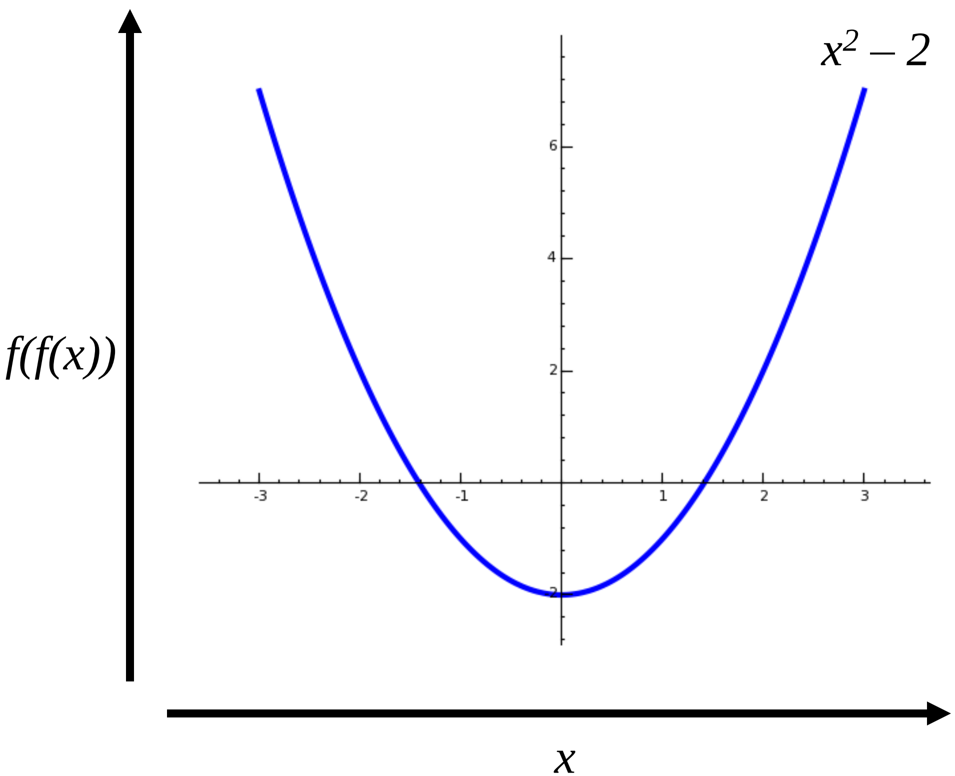
Yet, Genetic Programming can find these solutions easily….

$$f(f(x)) = x^2 - 2$$

Straightforward application of Symbolic Regression

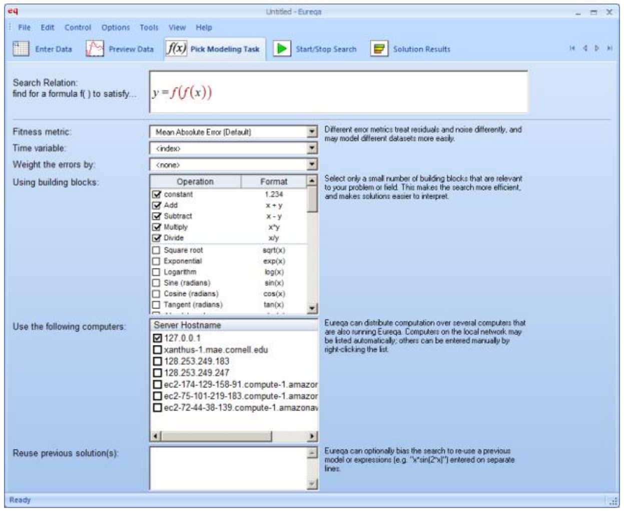
Fitness of a candidate $f(x)$ = $-\frac{1}{n}\sum_{i=1}^{n}\left[\, y_i - f\left(f\left(x_i\right)\right)\right]^2$

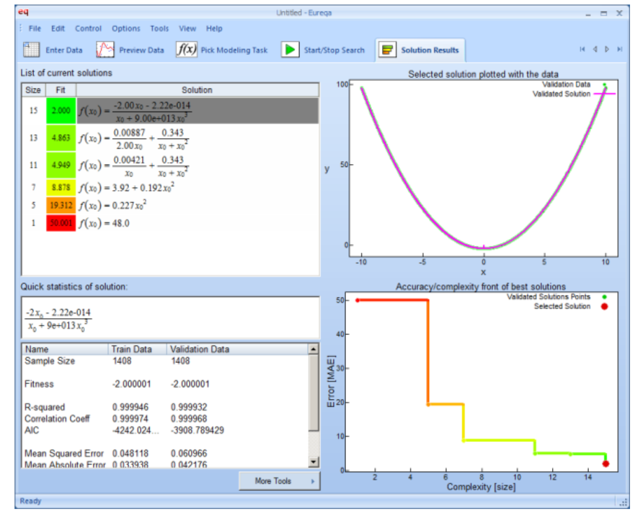
Solutions iterated twice:



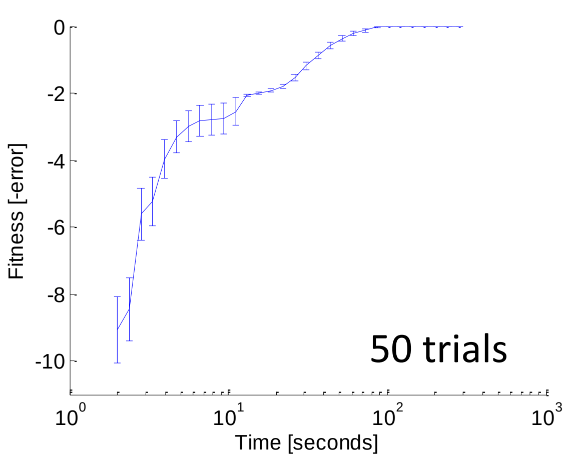
What is $f(x)$?

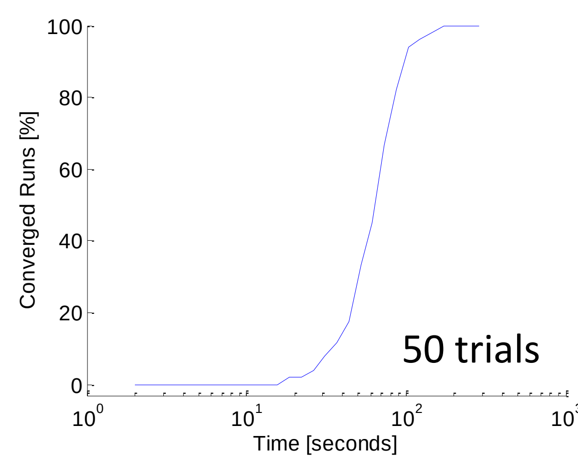
Solved in 81 seconds





And
Solved Reliably:





Nearly Perfect Fitness

$$f(x) = \frac{16.4916 - 2\cdot(\underline{1.16871\cdot10}^{18})\cdot x}{(\underline{1.16871\cdot10}^{18})\cdot x\cdot(16.4916 - 2\cdot(\underline{1.16871\cdot10}^{18})\cdot x^2)}$$

<u>The genetic program is trying to take a limit</u>....

$$f(x) = \lim_{a\to\infty} \frac{b - 2ax}{ax\left(b - 2ax^2\right)}$$

Exactly Correct Symbolicly

$$f\left(f(x)\right)=\frac{b-2a\left(f(x)\right)}{a\left(f(x)\right)\left(b-2a\left(f(x)\right)^{2}\right)}$$

$$\lim_{a\to\infty} f\left(f(x)\right)=\lim_{a\to\infty}\frac{b-2a\left(f(x)\right)}{a\left(f(x)\right)\left(b-2a\left(f(x)\right)^{2}\right)}$$

$$\lim_{a\to\infty} f\left(f(x)\right)=x^{2}-2$$

The solution is symbolicly correct

New Solution Found with Genetic Programming

$$f(f(x)) = x^2 - 2$$

$$f(x) = \lim_{a \to \infty} \frac{1 - 2ax}{ax\left(1 - 2ax^2\right)}$$

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The Best Entry:

- Entirely new solution found via GP
- Fastest this problem has ever been solved
- Potential impact in many fields, where such problems have never been solved before

Conclusions

Use GP to Solve Iterated Functions

$$f(f(x))$$